WASEDA University
Graduate School of Information, Production and Systems

Poster C6

Prime Factorization with Fewer Qubits by Combining Quantum Annealing and Classical Algorithm

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Adiabatic Quantum Computing Conference 2021 (AQC 2021)

June 23, 2021

Background: Prime Factorization

- For an input number N, find primes p and q satisfying

$$N = p \times q$$

  - Belong to NP problem, but generally not considered to be NP-complete [1]
- Recently, Ising computers have been developed and applied to combinatorial optimization problems
  - Many combinatorial optimization problems can be mapped into Ising model [2]
    - Prime Factorization is also considered for application
  - Ising model or QUBO (Quadratic Unconstrained Binary Optimization) formulation is the input of Ising computers
- QUBO is to decide the variable values minimizing
  - $O(\boldsymbol{x}) = \sum_i Q_{i,i} x_i + \sum_{i<j} Q_{i,j} x_i x_j \, , x_i \in \{0,1\}$
  - Formula of quadratic terms of binary variables

[1] Goldreich, O., & Wigderson, A. (2008). IV. 20 Computational Complexity. In The Princeton Companion to Mathematics (pp. 575-604). Princeton University Press.

[2] Lucas, A. (2014). Ising formulations of many NP problems. Frontiers in Physics, 2, 5.

Related Work: Multiplication Table Method [3]

- Binary multiplication is represented as addition of partial products
  - Addition of each column i should equal to N_i plus carry-outs
  - Carry variables are introduced column-wise
    - c_{ij} is generated at column i and affects to j
- MSBs and LSBs of p and q are set to 1
- Each column corresponds one term of $O(\boldsymbol{x})$
  - The square of the addition of each column terms plus carry-ins minus carry-outs and N_i, then sum
- Weakness
  - Requiring a lot of carry variables
  - Lots of auxiliary variables also needed to convert higher order terms to quadratic terms

Prime factorization problem

$N = p \times q$

p					1	p_2	p_1	1
q					1	q_2	q_1	1
$p \cdot q$					1	p_2	p_1	1
				q_1	p_2q_1	p_1q_1	q_1	
			q_2	p_2q_2	p_1q_2	q_2		
		1	p_2	p_1	1			
Carries	c_{67}	c_{56}	c_{45}	c_{34}	c_{23}	c_{12}		
	c_{57}	c_{46}	c_{35}	c_{24}				
N	1	0	0	0	1	1	1	1

Expanding, simplifying by $x^2 = x$, erasing high-order terms

$$
\begin{aligned}
&(p_1 + q_1 - 2c_{12} - 1)^2 + \\
&(p_2 + p_1q_1 + q_2 + c_{12} - (2c_{23} + 4c_{24}) - 1)^2 + \\
&(1 + p_2q_1 + p_1q_2 + 1 + c_{23} - (2c_{34} + 4c_{35}) - 1)^2 + \\
&(q_1 + p_2q_2 + p_1 + c_{24} + c_{34} - (2c_{45} + 4c_{46}))^2 + \\
&(p_2 + q_2 + c_{45} + c_{35} - (2c_{56} + 4c_{57}))^2 + \\
&(1 + c_{56} + c_{46} - 2c_{67})^2 + \\
&(c_{57} + c_{67} - 1)^2
\end{aligned}
$$

QUBO

[3] Schaller, G., & Schützhold, R. (2007). The role of symmetries in adiabatic quantum algorithms. arXiv preprint arXiv:0708.1882.

Related Work: Modified Multiplication Table Method [4]

- Several consecutive columns are merged into one block
  - 2 to 4 consecutive columns are combined
- Carries are generated block by block and $O(x)$ is also constructed from block by block
  - Carry variables are decided based on the former block
  - The number of carry variables can be reduced
- Factoring up to 376289 (19-bit number) using 94 logical qubits

Prime factorization problem

$N = p \times q$

Blocks	Ⅲ			Ⅱ		Ⅰ		
p					1	p_2	p_1	1
q					1	q_2	q_1	1
$p \cdot q$					1	p_2	p_1	1
				q_1	p_2q_1	p_1q_1	q_1	
			q_2	p_2q_2	p_1q_2	q_2		
		1	p_2	p_1	1			
$Carries$		c_4	c_3	c_2	c_1			
N	1	0	0	0	1	1	1	1

$$\left((p_1 + q_1) + 2(p_2 + p_1q_1 + q_2) - (8c_2 + 4c_1) - 3\right)^2 + \left((1 + p_2q_1 + p_1q_2 + 1 + c_1) + 2(q_1 + p_2q_2 + p_1 + c_2) - (8c_4 + 4c_3) - 1\right)^2 + \left((q_2 + p_2 + c_3) + 2(1 + c_4) - 4\right)^2$$

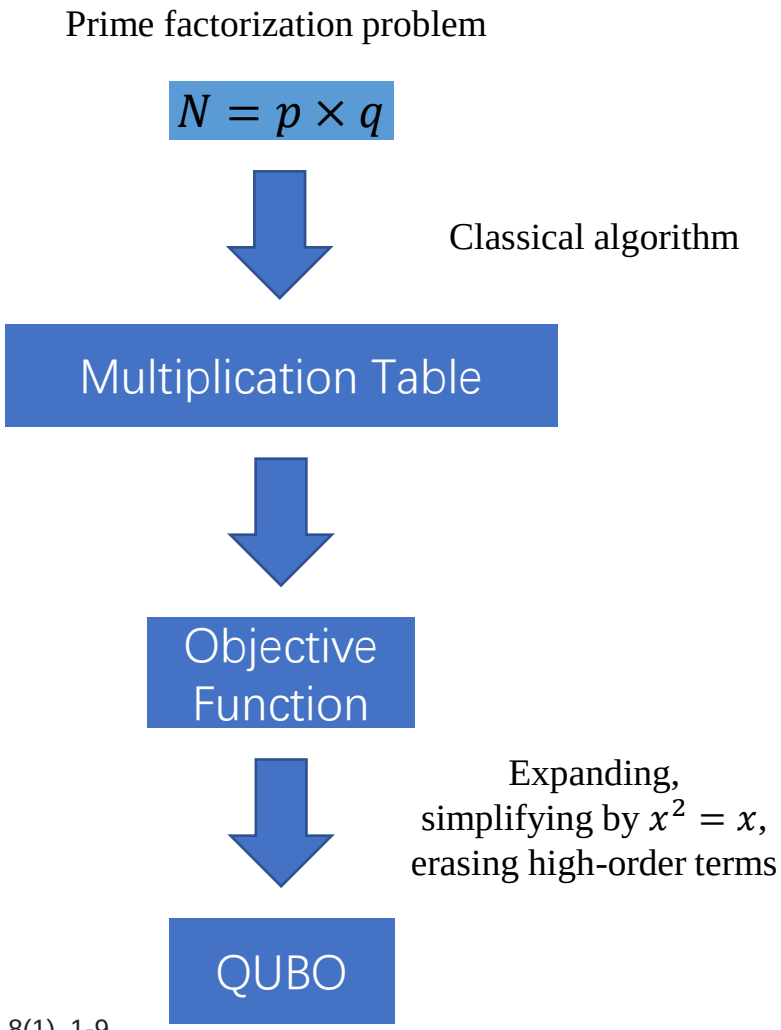
Expanding, simplifying by $x^2 = x$, erasing high-order terms

QUBO

[4] Jiang, S., Britt, K. A., McCaskey, A. J., Humble, T. S., & Kais, S. (2018). Quantum annealing for prime factorization. Scientific reports, 8(1), 1-9.

Related Work: Modified Multiplication Table Method with Optimized Parameters [5]

- Based on Modified Multiplication Method [4]
- To further reduce the number of carry variables
  - Employing a classical algorithm to generate carry variables, considering
    - The generation of carries is restricted by the target value
    - Carry-ins are usually smaller than assigning all carry variables to 1
- Factoring up to 1005973 (20-bit number) using 89 variables
- There needs a method to reduce variables for factoring larger numbers



[4] Jiang, S., Britt, K. A., McCaskey, A. J., Humble, T. S., & Kais, S. (2018). Quantum annealing for prime factorization. Scientific reports, 8(1), 1-9.

[5] Peng, W., Wang, B., Hu, F., Wang, Y., Fang, X., Chen, X., & Wang, C. (2019). Factoring larger integers with fewer qubits via quantum annealing with optimized parameters. SCIENCE CHINA Physics, Mechanics & Astronomy, 62(6), 60311.

Proposed Method: Multi-Step Quantum Annealing with Combination

- Construct QUBOs block by block and solve independently
  - Several groups of solutions obtained after annealing
- To obtain the correct solutions
  - Classical algorithm employed to combining several groups into one group
    - Same variables in the different groups should have same values

3 QUBOs are solved independently
(This is called as multi step)
3 solutions are obtained

Blocks	Ⅲ			Ⅱ		Ⅰ		
p					1	p_2	p_1	1
q					1	q_2	q_1	1
$p \cdot q$					1	p_2	p_1	1
				q_1	p_2q_1	p_1q_1	q_1	
			q_2	p_2q_2	p_1q_2	q_2		
		1	p_2	p_1	1			
$Carries$		c_4	c_3	c_2	c_1			
N	1	0	0	0	1	1	1	1

$$\left((q_2 + p_2 + c_3) + 2(1 + c_4) - 4\right)^2$$

$$\left((1 + p_2q_1 + p_1q_2 + 1 + c_1) + 2(q_1 + p_2q_2 + p_1 + c_2) - (8c_4 + 4c_3) - 1\right)^2$$

$$\left((p_1 + q_1) + 2(p_2 + p_1q_1 + q_2) - (8c_2 + 4c_1) - 3\right)^2$$

Implementation and Evaluation

- QUBO construction system is made based on
  - python 3.8.5 with
  - PyQUBO 0.4.0 [6] and
  - dwave-Qbsolv 0.3.1 [7]
- Dividing position
  - The position of the first bit of every block
- The data of Modified Multiplication Table Method and Modified Multiplication Table Method are from [5]

Instance	Dividing Position	Model	No. of variables
143 = 11 × 13	[1, 3, 5]	[4]	12
		[5]	5
		Proposal	11
221 = 13 × 17	[1, 4, 6]	[4]	15
		[5]	9
		Proposal	11
247 = 13 × 19	[1, 4, 6]	[4]	15
		[5]	10
		Proposal	11
323 = 17 × 19	[1, 4, 6]	[4]	20
		[5]	13
		Proposal	16
437 = 19 × 23	[1, 4, 6]	[4]	20
		[5]	13
		Proposal	16
589 = 19 × 31	[1, 4, 7]	[4]	19
		[5]	12
		Proposal	18
667 = 23 × 29	[1, 4, 6]	[4]	20
		[5]	12
		Proposal	17
899 = 29 × 31	[1, 4, 6]	[4]	20
		[5]	13
		Proposal	17

[4] Jiang, S., Britt, K. A., McCaskey, A. J., Humble, T. S., & Kais, S. (2018). Quantum annealing for prime factorization. Scientific reports, 8(1), 1-9.

[5] Peng, W., Wang, B., Hu, F., Wang, Y., Fang, X., Chen, X., & Wang, C. (2019). Factoring larger integers with fewer qubits via quantum annealing with optimized parameters. SCIENCE CHINA Physics, Mechanics & Astronomy, 62(6), 60311.

[6] Tanahashi, K., Takayanagi, S., Motohashi, T., & Tanaka, S. (2019). Application of Ising machines and a software development for Ising machines. Journal of the Physical Society of Japan, 88(6), 061010.

[7] http://github.com/dwavesystems/qbsolv

Experimental Result 2

Instance	Dividing Position	Model	No. of variables
989 = 23 × 43	[1, 4, 7]	[4]	24
		[5]	16
		Proposal	22
1073 = 29 × 37	[1, 4, 7]	[4]	24
		[5]	18
		Proposal	23
1591 = 37 × 43	[1, 4, 7]	[4]	29
		[5]	22
		Proposal	27
2449 = 31 × 79	[1, 4, 7]	[4]	28
		[5]	23
		Proposal	26
59989 = 239 × 251	[1, 4, 7, 10, 13]	[4]	59
		[5]	52
		Proposal	38
376289 = 571 × 659	[1, 5, 8, 11, 14, 17]	[4]	95
		[5]	90
		Proposal	56
1005973 = 997 × 1009	[1, 4, 7, 10, 13, 16]	[4]	96
		[5]	89
		Proposal	56

Variables can be reduced on large numbers

[4] Jiang, S., Britt, K. A., McCaskey, A. J., Humble, T. S., & Kais, S. (2018). Quantum annealing for prime factorization. Scientific reports, 8(1), 1-9.

[5] Peng, W., Wang, B., Hu, F., Wang, Y., Fang, X., Chen, X., & Wang, C. (2019). Factoring larger integers with fewer qubits via quantum annealing with optimized parameters. SCIENCE CHINA Physics, Mechanics & Astronomy, 62(6), 60311.

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[7] http://github.com/dwavesystems/qbsolv

Conclusions and Future Work

- A Multi-Step Quantum Annealing with Combination method is proposed
  - QUBOs are constructed block by block of modified multiplication table
  - Each QUBO is solved independently
  - Obtained solution sets are merged to one
- Proposed method is implemented in python and quantum annealing process is simulated using Qbsolv 0.3.1 provided by D-Wave
  - The number of variables for constructing QUBOs can be reduced for factorization of large numbers
- Merits
  - The number of variables of a QUBO can be reduced
- Demerits
  - Solutions of each QUBO might be large
- Future work
  - Reducing variables with reasonable number of solutions

Reference and Acknowledgement

[1] Goldreich, O., & Wigderson, A. (2008). IV. 20 Computational Complexity. In The Princeton Companion to Mathematics (pp. 575-604). Princeton University Press.

[2] Lucas, A. (2014). Ising formulations of many NP problems. Frontiers in Physics, 2, 5.

[3] Schaller, G., & Schützhold, R. (2007). The role of symmetries in adiabatic quantum algorithms. arXiv preprint arXiv:0708.1882.

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[7] http://github.com/dwavesystems/qbsolv

A part of this work was performed for Council for Science, Technology and Innovation (CSTI), Cross-ministerial Strategic Innovation Promotion Program (SIP), “Photonics and Quantum Technology for Society 5.0”(Funding agency : QST).