Adiabatic theorem for unbounded Hamiltonians *with cutoff*

of superconducting circuits

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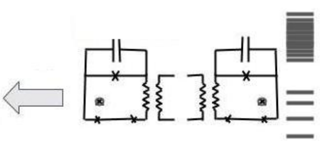
Overview

1. Example: Leakage for 1 qubit
2. Adiabatic theorem
3. Effective qubit Hamiltonian of a circuit

1-slide explanation

Given circuit controls $f_i^z(t)$, $f_i^x(t)$, we present a way to obtain:

$$H_{\mathrm{eff}}(t) = \sum J_{ij}(t) Z_i Z_j + h_i^z(t) Z_i + h_i^x(t) X_i$$



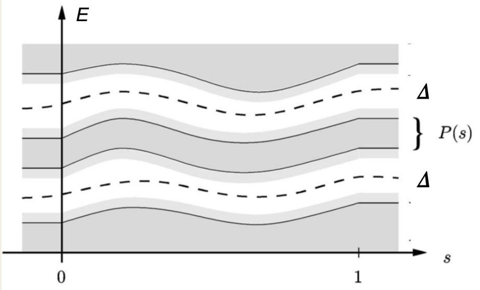
and compute the leakage out of qubit subspace.

Adiabatic theorem

If a subspace $P(t)\mathcal{H}$ of eigenstates of $H(t)$ is separated by a gap $\Delta(t)$ from the rest of the spectrum, then

$$\|(1-P(t))\psi(t)\| = O(1/t) \qquad (1)$$

where $\psi(t)$ is the solution of $\dot{\psi} = -iH(t)\psi$, $\psi(0) = P(0)\psi(0)$.



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$$\|(1 - P(t))\psi(t)\| = O(1/t) \tag{2}$$

where $\psi(t)$ is the solution of $\dot{\psi} = -iH(t)\psi$, $\psi(0) = P(0)\psi(0)$.

Big-O notation means $\exists\theta$:

$$\|(1 - P(t))\psi(t)\| \leq \theta/t \tag{3}$$

This talk is about the adiabatic timescale $\theta = \theta(H^0, \Delta \ldots)$

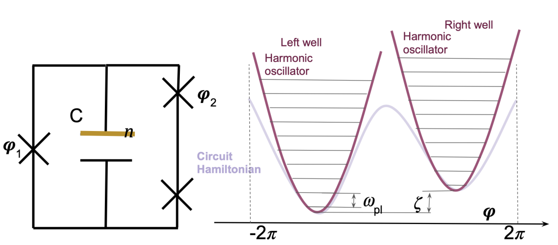
Example: capacitively-shunted flux qubit

$$H_{\mathrm{CSFQ,sin}} = E_C\hat{n}^2 + E_J b \cos\hat{\phi} \quad E_\alpha \sin\frac{1}{2}\hat{\phi}\sin\frac{1}{2}f \quad \phi\ \mathbf{2}\ [\quad 2\pi, 2\pi].$$

The $\hat{n}$ and $\hat{\phi}$ are canonically conjugate operators.

The E_J, E_C and E_α are fabrication parameters.

Time dependent controls $b(t)$ **2** $[1, \mathbf{B}]$, $f(t)$ **2** $[0, \pi/3]$.



Single-qubit anneal

$$H_{\text{CSFQ,sin}} = E_C \hat{n}^2 + E_J b \cos \hat{\phi} - E_\alpha \sin \frac{1}{2}\hat{\phi} \sin \frac{1}{2} f \quad \phi \in [-2\pi, 2\pi].$$

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- The E_J, E_C and E_α are fabrication parameters.
- Time dependent controls $b(t) \in [1, \mathbf{B}]$, $f(t) \in [0, \pi/3]$.

We follow the experimental procedure that aims at implementing:

$$H_\text{q} = h_x X + h_z Z = \omega_\text{q}((1 - s)X + sZ), \quad s = t/t_a \in [0, 1] \qquad (4)$$

The gap to the non-qubit states $\Delta \approx \omega_\text{pl} \approx \sqrt{E_J E_C b}$ (plasma frequency).

Our results

We present an explicit expression for θ^{new}[EM, D. Lidar, 2011.08116], improving on the existing result θ^{JRS} [S. Jansen, M.-B. Ruskai, and R. Seiler, (2007)] :

First practical application of both bounds to an anneal of a circuit model of a flux qubit

$$\omega_q \theta^{\text{JRS}} = \frac{11}{\sqrt{2}} \frac{\omega_q}{\omega_{\text{pl}}^{s=1} \delta} \tag{5}$$

the bound "leakage $< \theta/t$" is likely to be tight in this example

Hamiltonian

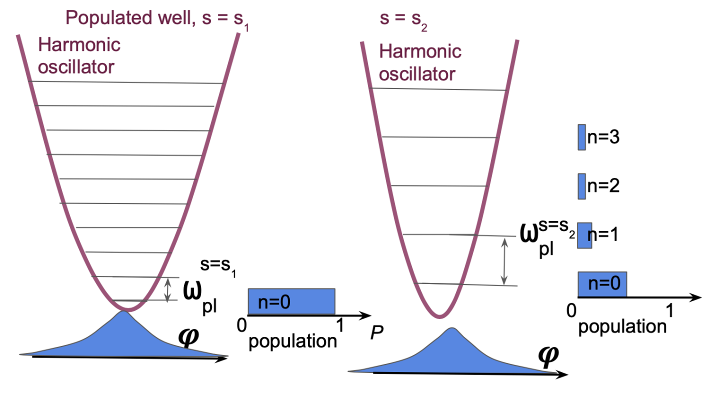
Note that we use a simpler model where the wells don't shift:

$$H_{\mathrm{CSFQ}} \mathbf{=} E_C \hat{n}^2 \mathbf{+} E_J b \cos\hat{\phi} \quad E_\alpha \cos\frac{1}{2}\mathbf{(}\hat{\phi} \quad f\mathbf{)} \quad \phi \ \mathbf{2}\ \mathbf{[} \quad 2\pi, 2\pi\mathbf{]}.$$

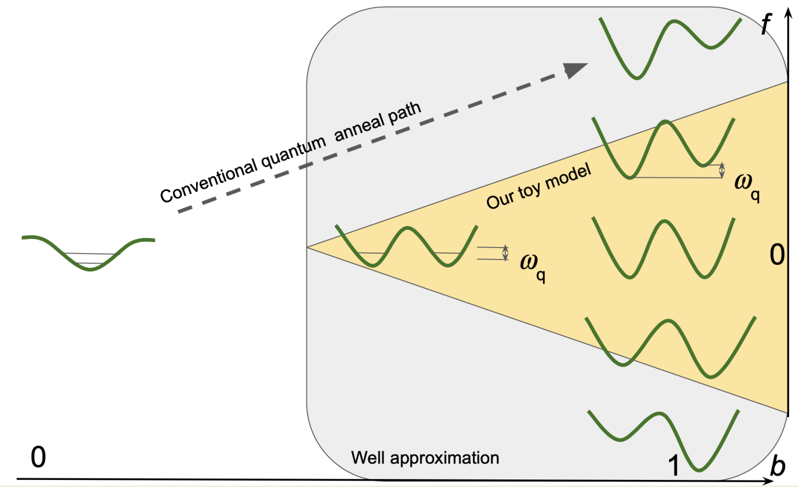
$$H_{\mathrm{CSFQ,sin}} \mathbf{=} E_C \hat{n}^2 \mathbf{+} E_J b \cos\hat{\phi} \quad E_\alpha \sin\frac{1}{2}\hat{\phi} \sin\frac{1}{2} f \quad \phi \ \mathbf{2}\ \mathbf{[} \quad 2\pi, 2\pi\mathbf{]}$$

Main source of leakage

Raising the barrier $b(t)$ leads to leakage:

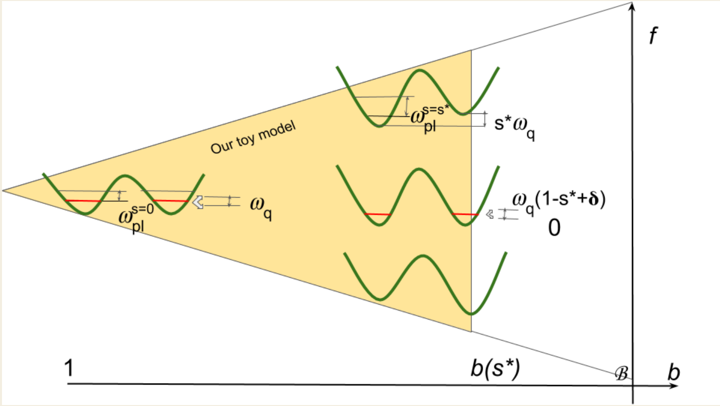


Well approximation



Validity of the qubit approximation $H_q = h_x X + h_z Z$

$$h_x^{s=s^\star} = \frac{1}{\sqrt{2}} \frac{\omega_q}{\omega_{pl}^{s=s^\star} t_a}, \quad \text{where} \quad h_x(s) = \omega_q(1-s) \qquad (6)$$

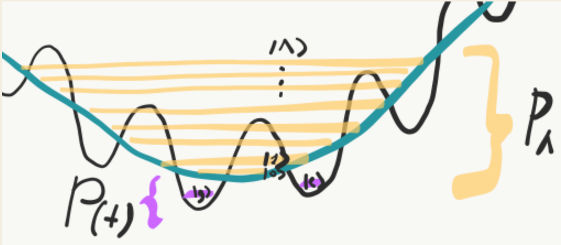


Here we always work with truncated H

Let $H(t)$ be a circuit Hamiltonian, for example:

$$H(t) = E_c n^2 + b(t) E_J \cos\phi + E_L(\phi + f_z(t))^2 \tag{7}$$

Using a basis $|m\rangle$ of $H_0 = E_c n^2 + E_L \phi^2$, define an isometry $V_\Lambda = \sum_{m=0}^{\Lambda} |e_m\rangle\langle m|$. The truncated Hamiltonian is: $H_{\text{tot}} = V_\Lambda H V_\Lambda^\dagger$



is a $\Lambda \times \Lambda$ matrix.

Our results

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- for an unbounded H with cuto□ Λ, $\mathbf{k}H^{\mathbf{0}}\mathbf{k}$, θ^{JRS} can be arbitrarily big, while θ^{new} is Λ-independent for relevant examples.
- for an n-qubit subspace $P(t)\,\mathbf{H}$, θ^{JRS} 2^n while θ^{new} 1

Unbounded H^0 (e.g. Harmonic oscillator)

For $\theta < \infty$, an assumption is needed.

Assumption of $\|R^0(z = i)H\| < \infty$ [J. E. Avron and A. Elgart, (1999), Nenciu (1980)]:, where the resolvent is:

$$R(z = i) = \frac{1}{i - H} = (i - H)^{-1} \tag{8}$$

No explicit $\theta(\|R^0(z = i)H\|, \dots)$ is presented.

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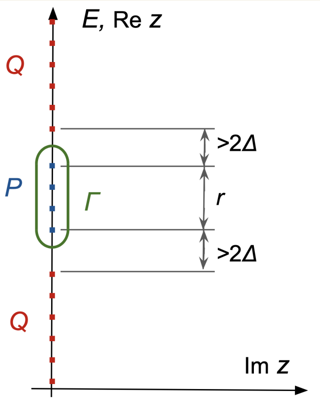
Our assumption: cuto□ Λ on H, H^{02} □ $\sum_{k=0}^{k_{m□x}} c_k H^{2k}$ *(easier to work with)*

Explicit $\theta(c_k, \ldots)$ is presented.

n-qubit low-energy subspace (e.g. D-wave)



A replacement

$$\frac{\sqrt{d}}{\Delta} \to \min\left(\frac{\sqrt{d}}{\Delta}, \frac{2r + 2\pi\Delta}{2\pi\Delta^2}\right)$$

can be made in θ^{JRS}, where $d = 2^n$ for n-qubit subspace $P\mathbf{H}$.

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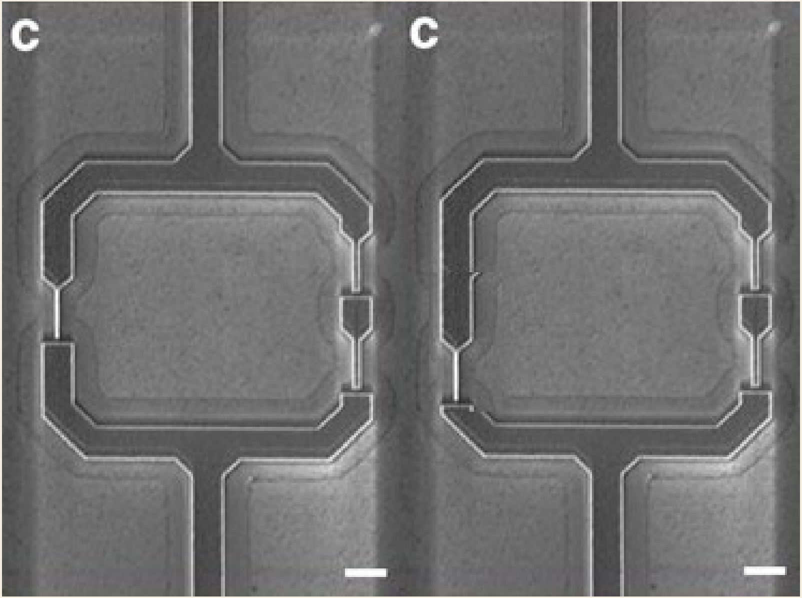
- both θ's can be applied to circuit design
- adiabatic theorem doubles as a method to calculate effective qubit Hamiltonian of a circuit

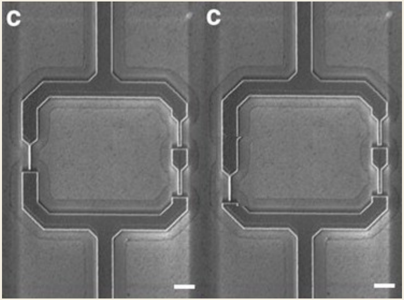
Junction positions - why not this:



Junction positions theory

$$H = E_C n^2 \quad 2E_J \cos(\phi \quad x f_t) + \alpha E_J \cos 2(\phi + (1 \quad x) f_t) \qquad (9)$$

Here x **2** $[0, 1]$ is a fraction of the total loop inductance corresponding to intervals between JJ.
□x_{opt} giving the best qubit approximation — least leakage to the third level along the annealing protocol.



["Flux qubit revisited…", 2016]□

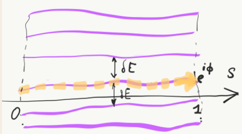
Effect of geometric terms on dynamics, reminder

$$|\dot\psi\rangle = -iH_q|\psi\rangle, \quad |\dot\psi_U\rangle = -i\left(U^\dagger H_q U + \frac{U^\dagger U'}{T}\right)|\psi_U\rangle, \quad |\dot\psi_G\rangle = -iU^\dagger H_q U|\psi_G\rangle$$

$\forall |\psi_G(0)\rangle = |\psi_U(0)\rangle$, nontrivial $H_q(t) = H_q(sT)$, smooth $U(s)$:

$$\left\| |\psi_U\rangle - |\psi_G\rangle \right\| = O\left(\left\|U'\right\|\right) = O(1) \qquad (10)$$

$$\left\| |\psi_U\rangle - e^{i\sum_\lambda \phi_\lambda |\lambda\rangle\langle\lambda|}|\psi_G\rangle \right\| \approx \frac{T_{ad}(\delta E)}{T} \gtrsim \text{at least } \frac{1}{\delta E T} \qquad (11)$$

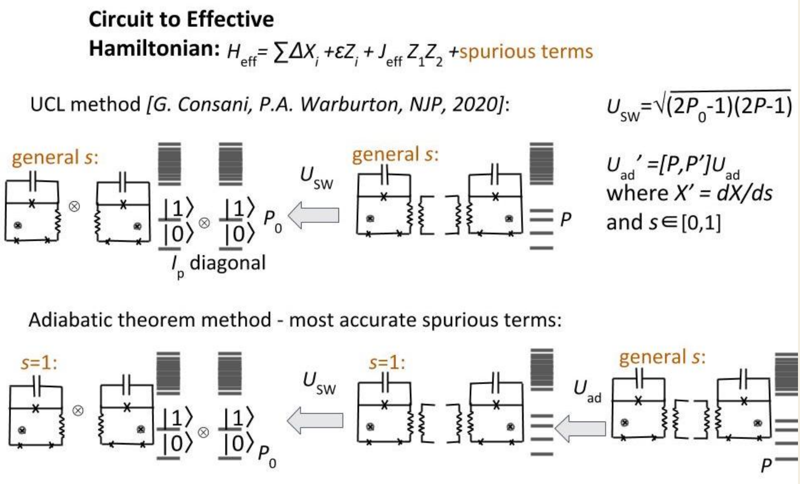


Eq. (11) has been observed by looking at final populations in different eigenstates. Here $\dot\psi = \frac{d\psi}{dt}$, $U' = \frac{dU}{ds}$.

Effective qubit Hamiltonian for a circuit $H(t)$

[EM, D. Lidar, 2020]: t-dependent controls $H(t)$ induce qubit dynamics:

(quantum annealing, $s = t/t_a$, $s \in [0, 1]$, no X terms at $s = 1$)



Effective Hamiltonian: problem statement

Given $|\psi(t)\rangle \in \mathcal{H}$ generated by $H(t)$, find $|\psi_{\text{eff}}(t)\rangle \in \mathcal{H}_{\text{eff}}$ generated by $H_{\text{eff}}(t)$, and an isometry $V(t) : \mathcal{H} \to \mathcal{H}_{\text{eff}}$ s.t.

$$\|V^\dagger|\psi_{\text{eff}}(t)\rangle - |\psi(t)\rangle\| \ll 1 \tag{12}$$

Let $\mathcal{H}_{\text{eff}}$ is the low-energy subspace of $H(t)$.

Effective Hamiltonian: result

Adiabatic theorem: Effective Hamiltonian:

$$\| U_{\mathrm{ad}}|\psi_0\rangle - |\psi(t)\rangle \| = O(1/T) \quad \rightarrow \quad \| U_{\mathrm{eff}}^\dagger |\psi_{\mathrm{eff}}(t)\rangle - |\psi(t)\rangle \| = O(1/T)$$

$$U'_{\mathrm{eff}} = [P', P]\, U_{\mathrm{eff}}, \; U_{\mathrm{eff}}(0) = \mathbb{1}$$

$$U'_{\mathrm{ad}} = (-iTH - [P', P])\, U_{\mathrm{ad}}, \; U_{\mathrm{ad}}(0) = \mathbb{1}$$

$$H_{\mathrm{eff}} = U_{\mathrm{eff}}^\dagger H U_{\mathrm{eff}} \tag{13}$$

Adiabatic theorem method, 1 qubit

Choose left and right well lowest energy states at $s = 1$ as $|0_1\rangle, |1_1\rangle$.

$H_{\text{tot}}(s), P(s)$ are a $\Lambda \times \Lambda$ matrices for the Hamiltonian and the projector on its 2 lowest eigenvalues.

Solve $U^0_{\text{eff}} = [P^0(s), P(s)]U_{\text{eff}}$, $U_{\text{eff}}(1) = \mathbf{1}$. A test $U_{\text{eff}}(s)P(s) = P(1)U_{\text{eff}}(s)$. At each step of the solution, compute the effective Hamiltonian, a 2×2 matrix

$$H^{ij}_{q,A.t.} = \langle i_1|U_{\text{eff}}(s)H_{\text{tot}}(s)U^{\dagger}_{\text{eff}}(s)|j_1\rangle, \quad i,j \in \{0,1\} \tag{14}$$

in the basis $U_{\text{eff}}(s)^{\dagger}|0_1\rangle, U_{\text{eff}}(s)^{\dagger}|1_1\rangle$ (Λ-dimensional vectors).

Define $|\dot{\psi}\rangle = -iH_{q,A.t.}|\psi\rangle$

Properties of the effective Hamiltonian

If obtained via the adiabatic theorem, the effective Hamiltonian is:

- Path-dependent.
- Schedule dependent in a sense that e.g. $H_{\text{tot},1}(s) = H_A + sH_B$ will have a different $H_{q,A.t.}$ from $H_{\text{tot},2}(s) = H_A + s^2 H_B$
- Anneal time T-independent: as long as $s = t/T$, the same effective Hamiltonian can describe evolution with different anneal times.